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Some properties of the fractional Ornstein–Uhlenbeck process*

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Abstract

We consider the fractional analogue of the Ornstein–Uhlenbeck process, i.e. the solution of the Langevin equation driven by a fractional Brownian motion in place of the usual Brownian motion. We establish some properties of these processes. We show that the process is local nondeterminism. For a two-dimensional process we show that its renormalized self-intersection local time exists in L^2 if and only if $0 < H < \frac{3}{4}$.

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1. Introduction

The Brownian motion and the Ornstein–Uhlenbeck process are the two most well-studied and widely applied stochastic processes. The Einstein–Smoluchowski theory may be seen as an idealized Ornstein–Uhlenbeck theory, and predictions of either cannot be distinguished by the experiment. However, if the Brownian particle is under the influence of an external force, the Einstein–Smoluchowski theory breaks down, while the Ornstein–Uhlenbeck theory remains successful. A diffusion process $X = \{X_t, 0 \leqslant t \leqslant T\}$ starting from $x \in \mathbb{R}$ is called the Ornstein–Uhlenbeck process with coefficient $\nu > 0$ if its infinitesimal generator is

$$L = \frac{1}{2}\nu^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x}.$$

The Ornstein–Uhlenbeck process (for the process, see, for example [21]) has a remarkable history in physics. It is used to model the velocity of the particle diffusion process and is the solution of the Langevin equation

$$\mathrm{d}X_t = -X_t\,\mathrm{d}t + \nu\,\mathrm{d}B_t, \qquad X_0 = x, \tag{1.1}$$

where B is a standard Brownian motion starting at zero.

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On the other hand, Brownian motion has been successfully extended to fractional Brownian motion (fBm), which has found wide applications in hydrology, condensed matter physics, biological physics, telecommunication networks, economics and other fields (see [4, 5, 11, 17, 28, 31] and references therein). A natural question arises whether there exists a generalized Ornstein–Uhlenbeck process which is related to fBm in a similar way as the Ornstein–Uhlenbeck process is linked to the Brownian motion. Lim and Muniandy [16] have introduced such an extension and studied some self-similar processes associated with this process. In [20] (see also [19]), Metzler–Klafter studied in detail the fractional Fokker–Planck equation and related topics, and in particular the fractional Fokker–Planck equation where the Ornstein–Uhlenbeck potential was also considered.

We now consider the Itô-type Langevin equation driven by the fractional Brownian motion

$$\mathrm{d}X_t^H = -X_t^H\,\mathrm{d}t + \nu\,\mathrm{d}B_t^H, \qquad X_0^H = x, \tag{1.2}$$

where $\nu > 0$ and $B^H = \{B_t^H, 0 \leqslant t \leqslant T\}$ is a fractional Brownian motion with the Hurst index $0 < H < 1$. Then equation (1.2) has a unique solution $X^H = \{X_t^H, 0 \leqslant t \leqslant T\}$, which can be expressed as

$$X_t^H = \mathrm{e}^{-t}\left(x + \nu\int_0^t \mathrm{e}^s\,\mathrm{d}B_s^H\right), \tag{1.3}$$

and the solution is called the *fractional Ornstein–Uhlenbeck process* (fractional O–U process). More work for the process can be found in [3, 15, 16, 20]. Clearly, when $H = 1/2$ the fractional O–U process is the classical Ornstein–Uhlenbeck process X with parameter $\nu > 0$ starting at $x \in \mathbb{R}$. It is well known that the classical Ornstein–Uhlenbeck process X can be obtained from Brownian motion by the so-called Lamperti transformation

$$X_t = x\,\mathrm{e}^{-t} + \frac{\nu}{\sqrt{2}}\mathrm{e}^{-t}B_{\mathrm{e}^{2t}-1}$$

in the sense of having the same finite dimensional distributions. However, for the fractional O–U process, Cheridito *et al* [3] (see also [16]) showed that X^H has not the same finite dimensional distribution as Lamperti transform

$$Z_t^H := x\,\mathrm{e}^{-t} + \nu\sqrt{H}\,\mathrm{e}^{-t}B^H_{\mathrm{e}^{t/H}-1}, \qquad t \geqslant 0 \tag{1.4}$$

if $H \neq 1/2$. Thus, it is an interesting problem to study some properties of the fractional O–U process. In the present paper, we shall consider some analytical properties of this process.

The structure of this paper is as follows. In section 2 we briefly recall the fractional Brownian motion and the related Wiener–Itô-type integral. In section 3 we show that the fractional O–U process X^H is local nondeterminism. The local nondeterminism means that there is an unremovable element of 'noise' in the local evolution of the sample function. We expect such a function to be 'locally irregular'. One of the major difficulties in studying the probabilistic, analytic or statistical properties of Gaussian random fields is the complexity of their dependence structures. As a result, many of the existing tools from theories on the Brownian motion, Markov processes or martingales fail for Gaussian random fields; and one often has to use general principles for Gaussian processes or to develop new tools. In many circumstances, the properties of local nondeterminism can help us to overcome this difficulty so that many elegant and deep results of Brownian motion (and Markov processes) can be extended to Gaussian (or stable) random fields. The concept of local nondeterminism of a Gaussian process was first introduced by Berman [1] to unify and extend his methods for studying the existence and joint continuity of local times of real-valued Gaussian processes. Berman's definition (see section 3) was later extended by Pitt [24] and Cuzick [7] to (N, d)-Gaussian random fields and by Cuzick [8] to local ϕ-nondeterminism for an arbitrary positive function ϕ. For more information on local nondeterminism, see Xiao [32], Nolan [22], Berman [2] and references therein. In section 4 we establish some estimates for the increments of the fractional O–U process. We show that there exist two constants $c_{H,T}, C_{H,T} > 0$ depending only on H, T such that the estimates

$$c_{H,T}\nu^2(t-s)^{2H} \leqslant E\big[\big(X_t^H - X_s^H\big)^2\big] \leqslant C_{H,T}\nu^2(t-s)^{2H},$$

$$c_{H,T}\nu^2[t^{2H}+s^{2H}-(t-s)^{2H}] \leqslant E\big[X_t^H X_s^H\big] \leqslant C_{H,T}\nu^2[t^{2H}+s^{2H}-(t-s)^{2H}]$$

hold for all $0 < s < t < T$, though the fractional O–U process is not stationary in general. In section 5 we consider the renormalized self-intersection local times of the fractional O–U process $\mathbf{X}^H$ in $\mathbb{R}^2$. The local time of self-intersections of Brownian motion appeared in the path integral to take into account the excluded volume effects in the study of polymer physics (Edwards [10]). It was also used in the study of quantum field theory by Symanzik [29]. In the appendix of that paper, Varadhan [30] rigorously discussed renormalizations of self-intersection local times for planar Brownian motion. Since then, many mathematical papers have appeared on this subject, for instance, by [9, 12, 14, 25, 26, 33] and references therein. The self-intersection local time measures the amount of time that the process spends intersecting itself on the time interval $[0, T]$ and has been an important topic of the theory of the stochastic process. A rigorous definition of this random variable may be obtained by approximating the Dirac function by the heat kernel

$$p_\varepsilon(x) = \frac{1}{2\pi\varepsilon}\,\mathrm{e}^{-\frac{|x|^2}{2\varepsilon}}, \qquad x \in \mathbb{R}^2, \quad \varepsilon > 0.$$

We denote the approximation by

$$\rho^{H,\varepsilon}(T) = \int_0^T\int_0^t p_\varepsilon\big(\mathbf{X}_t^H - \mathbf{X}_s^H\big)\,\mathrm{d}s\,\mathrm{d}t$$

and a natural question is to study the behavior of $\rho^{H,\varepsilon}(T)$ as ε tends to zero. We show that

$$\rho^{H,\varepsilon}(T) - E[\rho^{H,\varepsilon}(T)]$$

converges in L^2 as $\varepsilon \to 0$ if and only if $0 < H < \frac{3}{4}$, and the limit is called the renormalized self-intersection local time of $\mathbf{X}^H$. To avoid some lengthy calculations arising from proofs of theorems, we give an appendix which consists of estimates of some integrals. These estimates are of pure analysis.

2. Fractional Brownian motion

In this section, we briefly recall the definition and properties of the Wiener–Itô integral with respect to fBm. Throughout this paper we assume that $H \in \big(0, \frac{1}{2}\big) \cup \big(\frac{1}{2}, 1\big)$ is arbitrary but fixed and that $(\Omega, \mathcal{F}, \mu)$ is a complete probability space such that the fractional Brownian motion (fBm) B^H with the Hurst index H is well defined. For simplicity, we let C stand for a positive constant depending only on the subscripts and its value may be different in different appearances.

Recall that a centered continuous Gaussian process $B^H = \big\{B_t^H, 0 \leqslant t \leqslant T\big\}$ with the covariance function

$$E\big[B_t^H B_s^H\big] = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \qquad s, t \geqslant 0, \tag{2.1}$$

is called *the fractional Brownian motion with the Hurst index H*. This process was first introduced by Kolmogorov and studied by Mandelbrot and Van Ness [18], where a stochastic integral representation in terms of a standard Brownian motion was established.

If $H = 1/2$, B^H coincides with the standard Brownian motion B. Fractional Brownian motion has the following self-similar property: for any constant $a > 0$, the processes $\{a^{-H} B^H_{at}, t \geqslant 0\}$ and $\{B^H_t, t \geqslant 0\}$ have the same distribution. This property is an immediate consequence of the fact that the covariance function (2.1) is homogeneous of order $2H$, and is Hölder continuous of order $\alpha < H$. Set $X_n = B^H_n - B^H_{n-1}, n \geqslant 1$. Then $\{X_n, n \geqslant 1\}$ is a Gaussian stationary sequence with covariance function

$$\rho(n) = \tfrac{1}{2}[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}].$$

This yields $\rho_H(n) > 0$ (i.e. two increments of the form X_k and X_{n+k} are positively correlated) if $H > \frac{1}{2}$, and $\rho_H(n) > 0$ (i.e. they are negatively correlated) if $H < \frac{1}{2}$. This also implies that

$$\sum_{n=1}^{\infty} \rho(n) = \infty,$$

if $H > \frac{1}{2}$, i.e. the fractional Brownian motion B^H is *long-range dependence*, and

$$\sum_{n=1}^{\infty} |\rho(n)| < \infty.$$

These properties make B^H an interesting tool for many applications. However, we have known that fBm does not have independent increments, and B^H is neither a semi-martingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with B^H. More work for fractional Brownian motion can be found in Hu [13], Nualart [23], Coutin [6], Sebastian [27] and references therein.

Consider the integral representation of fBm B^H of the form

$$B^H_t = \int_0^t K_H(t,u)\,\mathrm{d}B_u, \qquad 0 \leqslant t \leqslant T, \tag{2.2}$$

where B is a standard Brownian motion and $K_H(t,u)$ is the kernel

$$K_H(t,u) = \kappa_H (t-u)^{H-\frac{1}{2}} + \kappa_H \left(\frac{1}{2} - H\right) \int_u^t (m-u)^{H-\frac{3}{2}} \left(1 - \left(\frac{u}{m}\right)^{\frac{1}{2}-H}\right) \mathrm{d}m \tag{2.3}$$

with a normalizing constant $\kappa_H > 0$ given by

$$\kappa_H = \left(\frac{2H\Gamma\left(\frac{3}{2} - H\right)}{\Gamma\left(H + \frac{1}{2}\right)\Gamma(2-2H)}\right)^{1/2}.$$

The kernel can be expressed as

$$\frac{\partial K_H}{\partial t}(t,u) = \kappa_H \left(\frac{1}{2} - H\right) \left(\frac{u}{t}\right)^{\frac{1}{2}-H} (t-u)^{H-\frac{3}{2}}, \tag{2.4}$$

and the kernel K defines an operator $\Gamma_{H,T}$ in $L^2([0;T])$ given by

$$\Gamma_{H,T} h(t) = \int_0^t K_H(t,u) h(u)\,\mathrm{d}u, \qquad h \in L^2([0;T])$$

and the function $\Gamma_{H,T} h(t)$ is continuous and vanishes at zero. The transpose $\Gamma^*_{H,t}$ of $\Gamma_{H,T}$ restricted to the interval $[0,t]$ $(0 \leqslant t \leqslant T)$ is

$$\Gamma^*_{H,T} g(u) = -\kappa_H u^{\frac{1}{2}-H} \frac{\mathrm{d}}{\mathrm{d}u} \int_u^t m^{H-\frac{1}{2}} (m-u)^{H-\frac{1}{2}} g(m)\,\mathrm{d}m, \qquad 0 \leqslant u \leqslant t$$

for $g \in \mathbf{S}$, the set of all smooth functions on $[0,T]$ with bounded derivatives. In particular, for $\frac{1}{2} < H < 1$, we have

$$\Gamma^*_{H,t} g(u) = \left(H - \frac{1}{2}\right) \kappa_H u^{\frac{1}{2}-H} \int_u^t m^{H-\frac{1}{2}} (m-u)^{H-\frac{3}{2}} g(m)\,\mathrm{d}m.$$

Now, we recall the definition of the Wiener–Itô integral with respect to fBm; more work can be found in Hu [13], Nualart [23] and references therein.

Definition 2.1. *Let*

$$\Theta_H = \left\{ f \in \mathbf{S} : \|f\| \equiv \int_0^T [\Gamma^*_{H,T} f(t)]^2 \, \mathrm{d}t < \infty \right\}.$$

For $f \in \Theta_H$ we define

$$\int_0^t f(u)\, \mathrm{d}B^H_u = \int_0^t \Gamma^*_{H,t} f(u)\, \mathrm{d}B_u, \qquad 0 \leqslant t \leqslant T,$$

where $B = \{B_t, 0 \leqslant t \leqslant T\}$ is a standard Brownian motion with $B_0 = 0$.

By applying the operator $\Gamma^*_{H,t}$, one can write the fractional O–U process $X^H = \left\{X^H_t, 0 \leqslant t \leqslant T\right\}$ starting from zero as

$$X^H_t = \nu \int_0^t F(t,u)\, \mathrm{d}B_u, \qquad 0 \leqslant t \leqslant T,$$

where B is a standard Brownian motion with $B_0 = 0$, and for $0 < u < t$

$$F(t,u) = \left(H - \frac{1}{2}\right) \kappa_H \,\mathrm{e}^{-t} u^{\frac{1}{2}-H} \int_u^t m^{H-\frac{1}{2}} (m-u)^{H-\frac{3}{2}} \,\mathrm{e}^m \, \mathrm{d}m \tag{2.5}$$

with $\frac{1}{2} < H < 1$, and

$$\begin{aligned} F(t,u) = \kappa_H u^{\frac{1}{2}-H} \Bigg(&-\mathrm{e}^{-t} \int_u^t (m-u)^{H-\frac{1}{2}} m^{H-\frac{1}{2}} \,\mathrm{e}^m \, \mathrm{d}m + t^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} \\ &+ \frac{2}{1-2H} \,\mathrm{e}^{-t} \int_u^t (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}} \,\mathrm{e}^m \, \mathrm{d}m \Bigg), \end{aligned} \tag{2.6}$$

with $0 < H < \frac{1}{2}$.

3. The local nondeterminism property

The main object of this section is to explain and prove theorem 3.1.

Theorem 3.1. *The fractional O–U process X^H is local nondeterministic.*

Thus, this theorem implies that there exists a constant $\gamma_0 > 0$ such that (see Berman [1]) the inequality

$$\mathrm{Var}\left(\sum_{j=2}^n u_j \big(X^H_{t_j} - X^H_{t_{j-1}}\big) \right) \geqslant \gamma_0 \sum_{j=2}^n u_j^2 \mathrm{Var}\big(X^H_{t_j} - X^H_{t_{j-1}}\big) \tag{3.1}$$

holds for $0 \leqslant t_1 < t_2 < \cdots < t_n \leqslant T$ and $u_j \in \mathbb{R}$, $j = 2, 3, \ldots, n$. Recall that a process $X = \{X_t, t \in J\}$ of the form

$$X_t = \int_0^t K(t,u)\, \mathrm{d}B_u, \qquad t \in J \tag{3.2}$$

is local nondeterministic (see Berman [1]) if and only if

$$\lim_{c \downarrow 0} \inf_{0 < t-s < c; s,t \in J} \frac{\int_s^t K^2(t,u)\, \mathrm{d}u}{\int_0^s [K(t,u) - K(s,u)]^2 \, \mathrm{d}u} > 0, \tag{3.3}$$

where F is a measurable function of (t,u) such that $\int_0^t K^2(t,u)\, \mathrm{d}u < \infty$ for all $t \in J$.

Proof of theorem 3.1. Consider the integral representation of the fractional O–U process X^H:

$$X_t^H = \nu \int_0^t F(t,u)\,\mathrm{d}B_u, \qquad 0 \leqslant t \leqslant T.$$

By lemma A.2 we only claim that

$$\int_s^t F^2(t,u)\,\mathrm{d}u \geqslant C_{H,T}(t-s)^{2H} \tag{3.4}$$

for all $0 < s < t$. For $\frac{1}{2} < H < 1$, by (2.5) we have

$$\begin{aligned} F(t,u) &\geqslant \left(H - \frac{1}{2}\right)\kappa_H\,\mathrm{e}^{-t+u}\int_u^t (m-u)^{H-\frac{3}{2}}\,\mathrm{d}m \\ &= \kappa_H\,\mathrm{e}^{-t+u}(t-u)^{H-\frac{1}{2}} > 0 \end{aligned}$$

for $0 < u < t$, which leads to

$$\int_s^t F^2(t,u)\,\mathrm{d}u \geqslant \kappa_H \mathrm{e}^{-2t+2s}(t-s)^{2H}. \tag{3.5}$$

On the other hand, let $0 < H < \frac{1}{2}$. Without loss of generality, one may assume $0 < s < t < 1$. We then have

$$F(t,u) \geqslant \kappa_H u^{\frac{1}{2}-H} t^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} > 0$$

for $0 < u < t$ by (2.6), which leads to

$$\begin{aligned} \int_s^t F^2(t,u)\,\mathrm{d}u &\geqslant (\kappa_H)^2 t^{2H-1}\int_s^t u^{1-2H}(t-u)^{2H-1}\,\mathrm{d}u \\ &\geqslant C_H(t-s)^{2H}. \end{aligned} \tag{3.6}$$

Combining these with lemma A.2, we show that the condition

$$\lim_{c\downarrow 0}\ \inf_{0<t-s<c;0<s,t\leqslant T} \frac{\int_s^t F^2(t,u)\,\mathrm{d}u}{\int_0^s [F(t,u)-F(s,u)]^2\,\mathrm{d}u} \geqslant C_{H,T} > 0 \tag{3.7}$$

is valid, which means that X^H is local nondeterministic for all $0 < H < 1$. This completes the proof. □

4. The variance of increments of the fractional O–U process

In this section, we will consider the variance of increments of the fractional O–U process. Let $X^H = \{X_t^H, 0 \leqslant t \leqslant T\}$ be a fractional O–U process starting from zero. Then we have

$$X_t^H = \nu \int_0^t \mathrm{e}^{-t+u}\,\mathrm{d}B_u^H = \nu \int_0^t F(t,u)\,\mathrm{d}B_u,$$

where B is a standard Brownian motion with $B_0 = 0$ and $F(t,u)$ is as in section 2, which implies that

$$E\left[X_t^H X_s^H\right] = \nu^2 \int_0^{t\wedge s} F(t,u)F(s,u)\,\mathrm{d}u.$$

In particular, for $\frac{1}{2} < H < 1$ we have

$$E\left[X_t^H X_s^H\right] = \nu^2\,\mathrm{e}^{-t-s}\int_0^t\int_0^s \mathrm{e}^{u+v}\phi(u,v)\,\mathrm{d}u\,\mathrm{d}v,$$

where $\phi(u, v) = H(2H-1)|u-v|^{2H-2}$. Thus, an elementary calculus can show that the estimates

$$\frac{1}{2}\nu^2 \mathrm{e}^{-t-s}[t^{2H}+s^{2H}-(t-s)^{2H}] \leqslant E\left[X_t^H X_s^H\right] \leqslant \frac{1}{2}\nu^2[t^{2H}+s^{2H}-(t-s)^{2H}]$$

hold for $\frac{1}{2} < H < 1$. More generally, we have the following proposition.

Proposition 4.1. *For* $0 < H < 1$, *we have*

$$c_{H,T}\nu^2[t^{2H}+s^{2H}-(t-s)^{2H}] \leqslant E\big[X_t^H X_s^H\big] \leqslant C_{H,T}\nu^2[t^{2H}+s^{2H}-(t-s)^{2H}] \tag{4.1}$$

for all $0 < s < t < T$.

Proof. Let $0 < H < \frac{1}{2}$. The left inequality in (4.1) follows from lemma A.4. In order to obtain the right inequality in (4.1), by (2.6) we have

$$\begin{aligned}
\kappa_H^{-2} E\big[X_t^H X_s^H\big] &= \kappa_H^{-2}\int_0^s F(t,u)F(s,u)\,\mathrm{d}u \\
&\leqslant \int_0^s u^{1-2H}\,\mathrm{d}u \int_u^t\int_u^s (mn)^{H-\frac{1}{2}}[(m-u)(n-u)]^{H-\frac{1}{2}}\,\mathrm{d}m\,\mathrm{d}n \\
&\quad + (st)^{H-\frac{1}{2}}\int_0^s u^{1-2H}[(t-u)(s-u)]^{H-\frac{1}{2}}\,\mathrm{d}u \\
&\quad + t^{H-\frac{1}{2}}\int_0^s u^{1-2H}(t-u)^{H-\frac{1}{2}}\,\mathrm{d}u \int_u^s (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}}\,\mathrm{d}m \\
&\quad + s^{H-\frac{1}{2}}\int_0^s u^{1-2H}(s-u)^{H-\frac{1}{2}}\,\mathrm{d}u \int_u^t (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}}\,\mathrm{d}m \\
&\quad + \int_0^s u^{1-2H}\,\mathrm{d}u \int_u^t\int_u^s (mn)^{H-\frac{3}{2}}[(m-u)(n-u)]^{H-\frac{1}{2}}\,\mathrm{d}m\,\mathrm{d}n.
\end{aligned}$$

Thus, lemma A.5 yields the right estimate in (4.1). This completes the proof. □

Theorem 4.2. *For* $0 \leqslant s < t < T$ *we set*

$$\sigma_{t,s}^2 = E\big[\big(X_t^H - X_s^H\big)^2\big], \tag{4.2}$$

then we have

$$\nu^2 c_{H,T}(t-s)^{2H} \leqslant \sigma_{t,s}^2 \leqslant \nu^2 C_{H,T}(t-s)^{2H}. \tag{4.3}$$

Proof. We have

$$\begin{aligned}
\sigma_{t,s}^2 &= \nu^2\int_0^t [F(t,u) - F(s,u)1_{[0,s]}(u)]^2\,\mathrm{d}u \\
&= \nu^2\int_0^s [F(t,u)-F(s,u)]^2\,\mathrm{d}u + \int_s^t F(t,u)^2\,\mathrm{d}u \\
&\geqslant \nu^2\int_s^t F(t,u)^2\,\mathrm{d}u
\end{aligned} \tag{4.4}$$

for all $0 < s < t < T$. Thus, the left inequality in (4.3) follows from (3.5) and (3.6). The right inequality in (4.3) follows from lemmas A.2 and A.3 and equality (4.4). This completes the proof. □

Let $\overline{\sigma} = E\big[\big(X_t^H - X_s^H\big)\big(X_{t'}^H - X_{s'}^H\big)\big]$ with $0 < s < t < T, 0 < s' < t' < T$ be the covariance between $X_t^H - X_s^H$ and $X_{t'}^H - X_{s'}^H$. Then the following result gives a characteristic of the expression

$$d_H(s,t,s',t') := \sigma_{s,t}^2\sigma_{s',t'}^2 - \overline{\sigma}^2.$$

Proposition 4.3.

(1) For $0 < s < s' < t < t' < T$*, we have*

$$d_H(s,t,s',t') \geqslant \kappa\nu^2[(t-s)^{2H}(t'-t)^{2H} + (t'-s')^{2H}(s'-s)^{2H}]. \tag{4.5}$$

(2) For $0 < s' < s < t < t' < T$*, we have*

$$d_H(s,t,s',t') \geqslant \kappa\nu^2(t-s)^{2H}(t'-s')^{2H}. \tag{4.6}$$

(3) For $0 < s < t < s' < t' < T$*, we have*

$$d_H(s,t,s',t') \geqslant \kappa\nu^2(t-s)^{2H}(t'-s')^{2H}, \tag{4.7}$$

where $\kappa > 0$ *is an enough small constant.*

One can show that proposition 4.3 holds by the same method as Hu [12]. The following result points out the fractional O–U process is not of long-range dependence.

Proposition 4.4. *Let*

$$\rho_H(n) = E\left[X_1^H\left(X_{n+1}^H - X_n^H\right)\right]$$

for every positive integer n. Then the series

$$\sum_{n=1}^{\infty} |\rho_H(n)|$$

converges for all $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$.

Proof. Let first $\frac{1}{2} < H < 1$. Clearly, we have

$$\mathrm{e}^{-1}\int_u^{n+1}(x-u)^{H-\frac{3}{2}}x^{H-\frac{1}{2}}\,\mathrm{e}^x\,\mathrm{d}x - \int_u^n (x-u)^{H-\frac{3}{2}}x^{H-\frac{1}{2}}\,\mathrm{e}^x\,\mathrm{d}x \sim n^{2H-3}\,\mathrm{e}^n \tag{4.8}$$

as $n \to \infty$. From (2.5) it follows that

$$|\rho_H(n)| = \nu^2\left|\int_0^{n+1} F(1,u)[F(n+1,u) - F(n,u)]\,\mathrm{d}u\right| \sim n^{2H-3} \qquad (n \to \infty),$$

which leads to $\sum_{n=1}^{\infty}|\rho_H(n)| < \infty$.

On the other hand, for $0 < H < \frac{1}{2}$ we have

$$\mathrm{e}^{-1}\int_u^{n+1}(x-u)^{H-\frac{1}{2}}x^{H-\frac{1}{2}}\,\mathrm{e}^x\,\mathrm{d}x - \int_u^n (x-u)^{H-\frac{1}{2}}x^{H-\frac{1}{2}}\,\mathrm{e}^x\,\mathrm{d}x \sim n^{2H-2}\,\mathrm{e}^n,$$

$$(n+1)^{H-\frac{1}{2}}(n+1-u)^{H-\frac{1}{2}} - n^{H-\frac{1}{2}}(n-u)^{H-\frac{1}{2}} \sim n^{2H-2}$$

as $n \to \infty$. Combining these with (4.8) and (2.6) leads to

$$\begin{aligned}|\rho_H(n)| &= \left|E\left[X_1^H\left(X_{n+1}^H - X_n^H\right)\right]\right| \\ &= \left|\nu^2\int_0^s F(1,u)[F(n+1,u) - F(n,u)]\,\mathrm{d}u\right| \\ &\leqslant C_H\nu^2 n^{2H-2},\end{aligned}$$

and so $\sum_{n=1}^{\infty}|\rho_H(n)| < \infty$. □

5. Renormalized self-intersection local time on $\mathbb{R}^2$

Let $\mathbf{X}^H = (X^{H,1}, X^{H,2})$ be a planar fractional Ornstein—Uhlenbeck process with the Hurst index $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ and $X_0^H = 0$, i.e., $X^{H,i} (i = 1, 2)$ is the unique solution of the Itô-type Langevin equation

$$\mathrm{d}X_t^{H,i} = -X_t^{H,i}\,\mathrm{d}t + \nu\,\mathrm{d}B_t^{H,i}, \qquad X_0^H = 0,$$

where $\nu > 0$ and $B^{H,1}, B^{H,2}$ are the two independent fractional Brownian motions with the Hurst index $0 < H < 1$. We define the self-intersection local time of X^H as

$$\rho^H(T) = \int_0^T\int_0^t \delta_0\left(\mathbf{X}_t^H - \mathbf{X}_s^H\right)\mathrm{d}t\,\mathrm{d}s,$$

where $\delta_0(x)$ is the Dirac delta function. It measures the amount of time that the process spends intersecting itself on the time interval $[0, T]$. A rigorous definition of this random variable may be obtained by approximating the Dirac function by the heat kernel

$$p_\varepsilon(x) = \frac{1}{2\pi\varepsilon}\,\mathrm{e}^{-\frac{|x|^2}{2\varepsilon}}, \qquad x \in \mathbb{R}^2, \qquad \varepsilon > 0.$$

We denote the approximation by

$$\rho^{H,\varepsilon}(T) = \int_0^T\int_0^t p_\varepsilon\left(\mathbf{X}_t^H - \mathbf{X}_s^H\right)\mathrm{d}s\,\mathrm{d}t$$

and a natural question is to study the behavior of $\rho^{H,\varepsilon}(T)$ as ε tends to zero. For $0 \leqslant s \leqslant t \leqslant T, 0 \leqslant s' \leqslant t' \leqslant T$ we denote

$$\sigma_{t,s}^2 = E\left[\left(X_t^{H,1} - X_s^{H,1}\right)^2\right], \qquad \overline{\sigma} = \mathrm{Cov}\left(X_t^{H,1} - X_s^{H,1}, X_{t'}^{H,1} - X_{s'}^{H,1}\right).$$

Using the following classical equality

$$p_\varepsilon(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\langle\xi,x\rangle}\,\mathrm{e}^{-\varepsilon|\xi|^2/2}\,\mathrm{d}\xi$$

from Fourier analysis, and the definition of $\rho^{H,\varepsilon}(T)$, we obtain

$$\rho^{H,\varepsilon}(T) = \frac{1}{(2\pi)^2}\int_0^T\int_0^t\int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\langle\xi,\mathbf{X}_t^H-\mathbf{X}_s^H\rangle}\,\mathrm{e}^{-\varepsilon\frac{|\xi|^2}{2}}\,\mathrm{d}\xi\mathrm{d}s\,\mathrm{d}t \tag{5.1}$$

Therefore, we get

$$\begin{aligned} E[\rho^{H,\varepsilon}(T)] &= \int_0^T\int_0^t E\left(p_\varepsilon\left(\mathbf{X}_t^H - \mathbf{X}_s^H\right)\right)\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^T\int_0^t \left(\varepsilon + \sigma_{t,s}^2\right)^{-1}\,\mathrm{d}s\,\mathrm{d}t, \end{aligned} \tag{5.2}$$

where we have used that $\left\langle\xi, \mathbf{X}_t^H - \mathbf{X}_s^H\right\rangle \sim N\left(0, |\xi|^2\sigma_{t,s}^2\right)$, so

$$E\left[\mathrm{e}^{\mathrm{i}\langle\xi,\mathbf{X}_t^H-\mathbf{X}_s^H\rangle}\right] = \mathrm{e}^{-\frac{1}{2}|\xi|^2\sigma_{t,s}^2},$$

and the fact that

$$\int_{\mathbb{R}^2} \mathrm{e}^{-\frac{1}{2}|\xi|^2(\varepsilon+\sigma^2(t,s))}\,\mathrm{d}\xi = \frac{2\pi}{\varepsilon + \sigma_{t,s}^2}.$$

According to representation (5.1), we get

$$E[\rho^{H,\varepsilon}(T)^2] = \frac{1}{(2\pi)^4}\int_{\mathbb{T}}\int_{\mathbb{R}^4} E\left(\mathrm{e}^{\mathrm{i}\langle\xi,\mathbf{X}_t^H-\mathbf{X}_s^H\rangle+\mathrm{i}\langle\eta,\mathbf{X}_{t'}^H-\mathbf{X}_{s'}^H\rangle}\right)\mathrm{e}^{-\varepsilon\frac{|\xi|^2+|\eta|^2}{2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}s'\,\mathrm{d}t',$$

where $\mathbb{T} = \{(s, t, s', t') : 0 < s < t < T, 0 < s' < t' < T\}$. Note that

$$\langle \xi, \mathbf{X}^H_t - \mathbf{X}^H_s \rangle + \langle \eta, \mathbf{X}^H_{t'} - \mathbf{X}^H_{s'} \rangle \sim N\left(0, |\xi|^2\sigma^2_{t,s} + |\eta|^2\sigma^2_{t',s'} + 2\overline{\sigma}\langle \xi, \eta\rangle\right)$$

for any $\xi, \eta \in \mathbb{R}^2$, we can write

$$\begin{aligned} E[\rho^{H,\varepsilon}(T)^2] &= \frac{1}{(2\pi)^4}\int_{\mathbb{T}}\int_{\mathbb{R}^4} \mathrm{e}^{-\frac{1}{2}((\sigma^2_{t,s}+\varepsilon)|\xi|^2+2\overline{\sigma}\langle\xi,\eta\rangle+(\sigma^2_{t',s'}+\varepsilon)|\eta|^2)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}s'\,\mathrm{d}t' \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{T}} \left(\left(\sigma^2_{t,s}+\varepsilon\right)\left(\sigma^2_{t',s'}+\varepsilon\right) - \overline{\sigma}^2\right)^{-1}\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}s'\,\mathrm{d}t' \end{aligned} \tag{5.3}$$

for all $\varepsilon > 0$. Combining this with (5.2), we get

$$\begin{aligned} E[\rho^{H,\varepsilon}(T)^2] - (E[\rho^{H,\varepsilon}(T)])^2 = \frac{1}{(2\pi)^2}\int_{\mathbb{T}} &\left[\left(\left(\sigma^2_{t,s}+\varepsilon\right)\left(\sigma^2_{t',s'}+\varepsilon\right) - \overline{\sigma}^2\right)^{-1}\right. \\ &\left. - \left(\left(\varepsilon+\sigma^2_{t,s}\right)\left(\varepsilon+\sigma^2_{t',s'}\right)\right)^{-1}\right]\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}s'\,\mathrm{d}t'. \end{aligned}$$

Lemma 5.1. *Let $\sigma^2_{t,s}$ and $\overline{\sigma}$ be defined as above. Then we have*

$$\int_{\mathbb{T}} \left[\left(\sigma^2_{t,s}\sigma^2_{t',s'} - \overline{\sigma}^2\right)^{-1} - \left(\sigma^2_{t,s}\sigma^2_{t',s'}\right)^{-1}\right]\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}s'\,\mathrm{d}t' < \infty$$

if and only if $0 < H < \frac{3}{4}$.

Proof. From Hu [12] and proposition 4.3 it is sufficient to show that the estimate

$$\overline{\sigma} = E\left[\left(X^{H,1}_t - X^{H,1}_s\right)\left(X^{H,1}_{t'} - X^{H,1}_{s'}\right)\right] \leqslant C_{H,T}\nu^2\mu(t, s, t', s') \tag{5.4}$$

holds for $0 < H < 1$ and $0 < s < t < T, 0 < s' < t' < T$, where

$$\mu(t, s, t', s') = \tfrac{1}{2}[|t' - s|^{2H} + |t - s'|^{2H} - |t - t'|^{2H} - |s' - s|^{2H}].$$

Without loss of generality, one may assume $t < t'$. For any $(s, t, s', t') \in \mathbb{T}$ we denote $\mathbb{T}_1 = \{0 \leqslant s < s' < t < t' \leqslant T\}, \mathbb{T}_2 = \{0 \leqslant s' < s < t < t' \leqslant T\}$ and $\mathbb{T}_3 = \{0 \leqslant s < t < s' < t' \leqslant T\}$. Let $0 < H < 1$. For $(s, t, s', t') \in \mathbb{T}_1$ we have

$$\begin{aligned} \overline{\sigma} &= \nu^2\int_0^t\int_0^{t'} [\mathrm{e}^{-t+u} - \mathrm{e}^{-s+u}1_{[0,s]}(u)][\mathrm{e}^{-t'+v} - \mathrm{e}^{-s'+v}1_{[0,s']}(v)]\phi(u, v)\,\mathrm{d}u\,\mathrm{d}v \\ &\leqslant \nu^2\int_0^s\int_0^{s'} [\mathrm{e}^{-t+u} - \mathrm{e}^{-s+u}][\mathrm{e}^{-t'+v} - \mathrm{e}^{-s'+v}]\phi(u, v)\,\mathrm{d}u\,\mathrm{d}v \\ &\quad + \int_s^t\int_{s'}^{t'} \mathrm{e}^{-t+u}\,\mathrm{e}^{-t'+v}\phi(u, v)\,\mathrm{d}u\,\mathrm{d}v \\ &\leqslant \nu^2(1 - \mathrm{e}^{-t+s})(1 - \mathrm{e}^{-t'+s'})\int_0^s\int_0^{s'} \phi(u, v)\,\mathrm{d}u\,\mathrm{d}v + \int_s^t\int_{s'}^{t'} \phi(u, v)\,\mathrm{d}u\,\mathrm{d}v \\ &\leqslant \nu^2(1 - \mathrm{e}^{-t+s'})^2(s'^{2H} - (s' - s)^{2H}) + (t' - s)^{2H} - (s' - s)^{2H} - (t' - t)^{2H} \\ &\leqslant \nu^2(T^{2H} \vee 1)(t - s')^{2H} + (t' - s)^{2H} - (s' - s)^{2H} - (t' - t)^{2H} \\ &\leqslant C_{T,H}\nu^2\mu(t, s, t', s'). \end{aligned}$$

Similarly, we can show that (5.4) holds for $(s, t, s', t') \in \mathbb{T}_i, i = 2, 3$. This completes the proof. □

Theorem 5.2. *The random variable $\rho^{H,\varepsilon}(T) - E[\rho^{H,\varepsilon}(T)]$ converges in L^2 as ε tends to zero if and only if $0 < H < \frac{3}{4}$. The limit is called the renormalized self-intersection local time of the fractional O–U process starting at zero.*

Proof. Clearly, as ε tends to zero $\rho^{H,\varepsilon}(T) - E[\rho^{H,\varepsilon}(T)]$ converges in L^2 if and only if

$$\mathrm{Var}(\rho^{H,\varepsilon}(T)) = E[\rho^{H,\varepsilon}(T)^2] - (E[\rho^{H,\varepsilon}(T)])^2$$

tends to a constant. Thus, the theorem follows from lemma 5.1. □

As a result of the theorem, we see that

$$\lim_{\varepsilon\to 0}[\rho^{H,\varepsilon}(T) - E\rho^{H,\varepsilon}(T)] = \infty$$

in L^2 if $H \geqslant \frac{3}{4}$, which implies that the renormalized self-intersection local time of the fractional O–U process does not exist for $H \geqslant \frac{3}{4}$. According to Hu and Nualart [14], one can conjecture that

$$\frac{1}{\sqrt{\log 1/\varepsilon}}[\rho^{H,\varepsilon}(T) - E\rho^{H,\varepsilon}(T)]$$

converges as ε tends to zero in distribution to a normal random variable for $H = \frac{3}{4}$.

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Appendix A. Estimates of some integrals

In order to obtain some estimates, we first give the following lemma (for the proof, see lemma 15.3 in [13]).

Lemma A.1. *Let* $\frac{1}{2} < H < 1, m, n > 0$. *Then we have*

$$\int_0^{m\wedge n} u^{1-2H}(m-u)^{H-\frac{3}{2}}(n-u)^{H-\frac{3}{2}}\,\mathrm{d}u = \alpha_H m^{\frac{1}{2}-H} n^{\frac{1}{2}-H}|m-n|^{2H-2},$$

where $m \wedge n = \min\{m, n\}$, $\alpha_H = B\left(2-2H, H-\frac{1}{2}\right)$ *and* $B(x, y)$ *is the beta function.*

Lemma A.2. *Let* $F(\cdot,\cdot)$ *be given by (2.5) and (2.6). Then we have*

$$\int_0^s [F(t,u) - F(s,u)]^2\,\mathrm{d}u \leqslant C_{H,T}(t-s)^{2H}, \qquad t \geqslant s \geqslant 0, \tag{A.1}$$

for all $0 < H < 1$.

Proof. The proof will be done in two cases.

First, let us consider $\frac{1}{2} < H < 1$. By (2.5) we have, for all $0 < s < t < T$

$$\begin{aligned}|F(t,u) - F(s,u)| &\leqslant \kappa_H u^{\frac{1}{2}-H}|\mathrm{e}^{-t} - \mathrm{e}^{-s}|\int_u^s m^{H-\frac{1}{2}}(m-u)^{H-\frac{3}{2}}\,\mathrm{e}^m\,\mathrm{d}m \\ &\quad + \mathrm{e}^{-t}\kappa_H u^{\frac{1}{2}-H}\int_s^t m^{H-\frac{1}{2}}(m-u)^{H-\frac{3}{2}}\,\mathrm{e}^m\,\mathrm{d}m \\ &\equiv \kappa_H u^{\frac{1}{2}-H}[A_1 + A_2].\end{aligned}$$

But, for all $0 < u < s < t < T$ we have $A_2 \leqslant \int_s^t m^{H-\frac{1}{2}}(m-u)^{H-\frac{3}{2}}\,\mathrm{d}m$ and

$$\begin{aligned}
A_1 &= |\mathrm{e}^{-t} - \mathrm{e}^{-s}| \int_u^s m^{H-\frac{1}{2}}(m-u)^{H-\frac{3}{2}}\,\mathrm{e}^m\,\mathrm{d}m \\
&\leqslant (t-s)\int_u^s m^{H-\frac{1}{2}}(m-u)^{H-\frac{3}{2}}\,\mathrm{d}m \\
&\leqslant \frac{2}{2H-1}(t-s)s^{H-\frac{1}{2}}(s-u)^{H-\frac{1}{2}}.
\end{aligned}$$

It follows from lemma A.1 that

$$\begin{aligned}
\int_0^s [F(t,u) - F(s,u)]^2\,\mathrm{d}u &\leqslant 2(\kappa_H)^2 \int_0^s u^{1-2H} A_1^2\,\mathrm{d}u + 2(\kappa_H)^2 \int_0^s u^{1-2H} A_2^2\,\mathrm{d}u \\
&\leqslant C_H s^{2H}(t-s)^2 \\
&\quad + 2(\kappa_H)^2 \int_s^t\int_s^t m^{H-\frac{1}{2}} n^{H-\frac{1}{2}}\,\mathrm{d}m\,\mathrm{d}n \int_0^s u^{1-2H}(m-u)^{H-\frac{3}{2}}(n-u)^{H-\frac{1}{2}}\,\mathrm{d}u \\
&\leqslant C_H s^{2H}(t-s)^2 \\
&\quad + 2(\kappa_H)^2 \int_s^t\int_s^t m^{H-\frac{1}{2}} n^{H-\frac{1}{2}}\,\mathrm{d}m\,\mathrm{d}n \int_0^{m\wedge n} u^{1-2H}(m-u)^{H-\frac{3}{2}}(n-u)^{H-\frac{1}{2}}\,\mathrm{d}u \\
&\leqslant C_H s^{2H}(t-s)^2 + 2C_H \int_s^t\int_s^t |m-n|^{2H-2}\,\mathrm{d}m\,\mathrm{d}n \\
&\leqslant C_H s^{2H}(t-s)^2 + 2C_H (t-s)^{2H} \\
&\leqslant T^2 C_H (t-s)^{2H}.
\end{aligned}$$

Next, let us consider the case $0 < H < \frac{1}{2}$. Without loss of generality one may assume $0 < s < t < 1$. We will obtain the estimate in three steps. For all $0 < s < t < 1$ we have

$$F(t,u) - F(s,u) = \kappa_H u^{\frac{1}{2}-H}(B_1 + B_2 + B_3),$$

where

$$B_1 := t^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}(s-u)^{H-\frac{1}{2}},$$

$$B_2 := \mathrm{e}^{-s}\int_u^s (m-u)^{H-\frac{1}{2}} m^{H-\frac{1}{2}}\,\mathrm{e}^m\,\mathrm{d}m - \mathrm{e}^{-t}\int_u^t (m-u)^{H-\frac{1}{2}} m^{H-\frac{1}{2}}\,\mathrm{e}^m\,\mathrm{d}m,$$

$$\begin{aligned}
B_3 := &\frac{2}{1-2H}\,\mathrm{e}^{-t}\int_u^t (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}}\,\mathrm{e}^m\,\mathrm{d}m \\
&- \frac{2}{1-2H}\,\mathrm{e}^{-s}\int_u^s (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}}\,\mathrm{e}^m\,\mathrm{d}m \\
= &\frac{2}{1-2H}(\mathrm{e}^{-t} - \mathrm{e}^{-s})\int_u^s (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}}\,\mathrm{e}^m\,\mathrm{d}m \\
&+ \frac{2}{1-2H}\,\mathrm{e}^{-t}\int_s^t (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}}\,\mathrm{e}^m\,\mathrm{d}m.
\end{aligned}$$

Step I. We claim that

$$\int_0^s u^{1-2H}|B_3|^2\,\mathrm{d}u \leqslant C_{H,T}(t-s)^{2H}.$$

We have

$$
\begin{aligned}
|B_3| &\leqslant \frac{2}{1-2H}\left[(t-s)\int_u^s (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}}\,\mathrm{d}m + \int_s^t m^{H-\frac{3}{2}}(m-u)^{H-\frac{1}{2}}\,\mathrm{d}m\right] \\
&= \frac{2}{1-2H}\left[(t-s)\int_u^s (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}}\,\mathrm{d}m + \int_s^t m^{H-\frac{3}{2}}(m-u)^{H-\frac{1}{2}}\,\mathrm{d}m\right] \\
&\equiv \frac{2}{1-2H}[(t-s)B_{3,1} + B_{3,2}].
\end{aligned}
$$

But by lemma A.1 we have

$$
\begin{aligned}
\int_0^s u^{1-2H} B_{3,1}^2\,\mathrm{d}u &= \int_0^s u^{1-2H}\,\mathrm{d}u \int_u^s \int_u^s [(m-u)(n-u)]^{H-\frac{1}{2}} (mn)^{H-\frac{3}{2}}\,\mathrm{d}m\,\mathrm{d}n \\
&= \int_0^s \int_0^s (mn)^{H-\frac{3}{2}}\,\mathrm{d}m\,\mathrm{d}n \int_0^{m\wedge n} u^{1-2H}[(m-u)(n-u)]^{H-\frac{1}{2}}\,\mathrm{d}u \\
&\leqslant \int_0^s \int_0^s (mn)^{H-\frac{1}{2}}\,\mathrm{d}m\,\mathrm{d}n \int_0^{m\wedge n} u^{-2H}[(m-u)(n-u)]^{H-1}\,\mathrm{d}u \\
&\equiv \int_0^s \int_0^s (mn)^{H-\frac{1}{2}}\,\mathrm{d}m\,\mathrm{d}n \int_0^{m\wedge n} u^{1-2(H+\frac{1}{2})}[(m-u)(n-u)]^{(H+\frac{1}{2})-\frac{3}{2}}\,\mathrm{d}u \\
&= \alpha_H \int_0^s \int_0^s (mn)^{-\frac{1}{2}} |m-n|^{2H-1}\,\mathrm{d}m\,\mathrm{d}n \\
&= \frac{2\alpha_H}{H} s^{2H} \int_0^1 x^{-\frac{1}{2}} (1-x)^{2H-1}\,\mathrm{d}x,
\end{aligned}
\tag{A.2}
$$

where we have used the facts $u \leqslant (mn)^{\frac{1}{2}}$, $(m-u)(n-u) < mn$. Similarly, one can get

$$
\begin{aligned}
\int_0^s u^{1-2H} B_{3,2}^2\,\mathrm{d}u &= \int_0^s u^{1-2H}\,\mathrm{d}u \int_s^t \int_s^t [(m-u)(n-u)]^{H-\frac{1}{2}} (mn)^{H-\frac{3}{2}}\,\mathrm{d}m\,\mathrm{d}n \\
&\leqslant \int_s^t \int_s^t (mn)^{H-\frac{3}{2}}\,\mathrm{d}m\,\mathrm{d}n \int_0^{m\wedge n} u^{1-2H}[(m-u)(n-u)]^{H-\frac{1}{2}}\,\mathrm{d}u \\
&\leqslant \frac{2\alpha_H}{H}(t-s)^{2H} \int_0^1 x^{-\frac{1}{2}}(1-x)^{2H-1}\,\mathrm{d}x.
\end{aligned}
$$

Thus, we obtain

$$
\int_0^s u^{1-2H}|B_3|^2\,\mathrm{d}u \leqslant C_H[(t-s)^2 s^{2H} + (t-s)^{2H}] \leqslant C_{H,T}(t-s)^{2H}.
$$

Step II. We claim that

$$
\int_0^s u^{1-2H}|B_2|^2\,\mathrm{d}u \leqslant C_{H,T}(t-s)^{2H}.
$$

Noting that

$$
\begin{aligned}
|B_2| &\leqslant (\mathrm{e}^{-s} - \mathrm{e}^{-t}) \int_u^s (m-u)^{H-\frac{1}{2}} m^{H-\frac{1}{2}} \mathrm{e}^m\,\mathrm{d}m + \mathrm{e}^{-t}\int_s^t (m-u)^{H-\frac{1}{2}} m^{H-\frac{1}{2}} \mathrm{e}^m\,\mathrm{d}m \\
&\leqslant (t-s)\int_u^s (m-u)^{H-\frac{1}{2}} m^{H-\frac{1}{2}}\,\mathrm{d}m + \int_s^t (m-u)^{H-\frac{1}{2}} m^{H-\frac{1}{2}}\,\mathrm{d}m \\
&\leqslant \frac{2}{2H+1} u^{H-\frac{1}{2}}\big[(t-s)(s-u)^{H+\frac{1}{2}} + t^{H+\frac{1}{2}} - s^{H+\frac{1}{2}}\big] \\
&\leqslant \frac{2}{2H+1} u^{H-\frac{1}{2}}\big[(t-s)(s-u)^{H+\frac{1}{2}} + (t-s)^{H+\frac{1}{2}}\big],
\end{aligned}
$$

we get

$$\int_0^s u^{1-2H}|B_2|^2\,\mathrm{d}u \leqslant C_H[s^{2H+2}(t-s)^2 + s(t-s)^{2H+1}] \leqslant C_{H,T}(t-s)^{2H}.$$

Step III. We finally claim that

$$\int_0^s u^{1-2H}|B_1|^2\,\mathrm{d}u \leqslant C_{H,T}(t-s)^{2H}.$$

This is clear. Thus, we have proved

$$\begin{aligned}\int_0^s [F(t,u)-F(s,u)]^2\,\mathrm{d}u &\leqslant 3\kappa_H^2\int_0^t u^{1-2H}(|B_1|^2+|B_2|^2+|B_3|^2)\,\mathrm{d}u\\ &\leqslant C_{H,T}(t-s)^{2H}.\end{aligned}$$

This completes the proof. □

Lemma A.3. *Let $F(\cdot,\cdot)$ be given by (2.5) and (2.6). Then the estimate*

$$\int_s^t F(t,u)^2\,\mathrm{d}u \leqslant C_{H,T}(t-s)^{2H}, \qquad t\geqslant s\geqslant 0, \tag{A.3}$$

holds for all $0<H<1$.

Proof. For $\frac{1}{2}<H<1$, by lemma A.1 we have

$$\begin{aligned}&\int_s^t F(t,u)^2\,\mathrm{d}u = C_H\,\mathrm{e}^{-2t}\int_s^t u^{1-2H}\,\mathrm{d}u\,\cdot\\ &\int_u^t\int_u^t (mn)^{H-\frac{1}{2}}[(m-u)(n-u)]^{H-\frac{3}{2}}\,\mathrm{e}^{m+n}\,\mathrm{d}m\,\mathrm{d}n\\ &\qquad\leqslant C_H\int_s^t\int_s^t (mn)^{H-\frac{1}{2}}\,\mathrm{d}m\,\mathrm{d}n\int_s^{m\wedge n} u^{1-2H}[(m-u)(n-u)]^{H-\frac{3}{2}}\,\mathrm{d}u\\ &\qquad\leqslant C_H\int_s^t\int_s^t |m-n|^{2H-2}\,\mathrm{d}m\,\mathrm{d}n\\ &\qquad= C_H(t-s)^{2H}.\end{aligned}$$

For $0<H<\frac{1}{2}$, by (2.6) we have

$$\begin{aligned}|F(t,u)| &\leqslant C_H\left(u^{\frac{1}{2}-H}\int_u^t (m-u)^{H-\frac{1}{2}}m^{H-\frac{1}{2}}\,\mathrm{d}m + u^{\frac{1}{2}-H}t^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}}\right.\\ &\qquad\left.+u^{\frac{1}{2}-H}\int_u^t (m-u)^{H-\frac{1}{2}}m^{H-\frac{3}{2}}\,\mathrm{d}m\right)\\ &\equiv C_H(\mathrm{I}_{t,u}+\mathrm{II}_{t,u}+\mathrm{III}_{t,u})\end{aligned}$$

with $0<u<t$. But, for $0<u<t<T$ we have

$$\int_s^t \mathrm{I}_{t,u}^2\,\mathrm{d}u \leqslant \int_s^t du\left(\int_u^t (m-u)^{H-\frac{1}{2}}\,\mathrm{d}m\right)^2 = C_H(t-s)^{2H+2},$$

$$\int_s^t \mathrm{II}_{t,u}^2\,\mathrm{d}u \leqslant \int_s^t u^{1-2H}t^{2H-1}(t-u)^{2H-1}\,\mathrm{d}u \leqslant \frac{1}{2H}(t-s)^{2H},$$

and by lemma A.1 and $u \leqslant (mn)^{\frac{1}{2}}$, $(m-u)(n-u) < mn$

$$\begin{aligned}\int_s^t \mathrm{III}_{t,u}^2\,\mathrm{d}u &= \int_s^t u^{1-2H}\,\mathrm{d}u \int_u^t\int_u^t [(m-u)(n-u)]^{H-\frac{1}{2}}(mn)^{H-\frac{3}{2}}\,\mathrm{d}m\,\mathrm{d}n \\ &= \int_s^t\int_s^t (mn)^{H-\frac{3}{2}}\,\mathrm{d}m\,\mathrm{d}n \int_s^{m\wedge n} u^{1-2H}[(m-u)(n-u)]^{H-\frac{1}{2}}\,\mathrm{d}u \\ &\leqslant \int_s^t\int_s^t (mn)^{H-\frac{1}{2}}\,\mathrm{d}m\,\mathrm{d}n \int_0^{m\wedge n} u^{-2H}[(m-u)(n-u)]^{H-1}\,\mathrm{d}u \\ &\equiv \int_s^t\int_s^t (mn)^{H-\frac{1}{2}}\,\mathrm{d}m\,\mathrm{d}n \int_0^{m\wedge n} u^{1-2(H+\frac{1}{2})}[(m-u)(n-u)]^{(H+\frac{1}{2})-\frac{3}{2}}\,\mathrm{d}u \\ &= \alpha_H \int_s^t\int_s^t (mn)^{-\frac{1}{2}}|m-n|^{2H-1}\,\mathrm{d}m\,\mathrm{d}n \\ &\leqslant C_H(t-s)^{2H},\end{aligned}$$

leads to

$$\int_s^t F(t,u)^2\,\mathrm{d}u \leqslant C_{T,H}(t-s)^{2H}.$$

This completes the proof. □

Lemma A.4. *Let* $0 < H < \frac{1}{2}$. *Then the estimate*

$$\int_0^s F(t,u)F(s,u)\,\mathrm{d}u \geqslant C_{H,T}[t^{2H} + s^{2H} - (t-s)^{2H}] \tag{A.4}$$

holds for $0 < s \leqslant t$.

Proof. Given $0 < H < \frac{1}{2}$. Without loss of generality, one may assume $0 < s < t < 1$. It follows from (2.6) that

$$F(t,u) \geqslant \int_u^t (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}}\,\mathrm{d}m$$

for $0 < u < t$, which implies that by $(m-u)^{H-\frac{1}{2}} \geqslant m^{H-\frac{1}{2}}$ $(0 < u < m)$

$$\begin{aligned}\int_0^s F(t,u)F(s,u)\,\mathrm{d}u &\geqslant C_H\int_0^s u^{1-2H}\,\mathrm{d}u\int_u^t\int_u^s [(m-u)(n-u)]^{H-\frac{1}{2}}(nm)^{H-\frac{3}{2}}dndm \\ &\geqslant C_H\int_0^s\int_0^s (mn)^{H-\frac{3}{2}}dndm\int_0^{m\wedge n} u^{1-2H}[(m-u)(n-u)]^{H-\frac{1}{2}}\,\mathrm{d}u \\ &\geqslant C_H\int_0^s\int_0^m (mn)^{H-\frac{3}{2}}dndm\int_0^n u^{1-2H}[(m-u)(n-u)]^{H-\frac{1}{2}}\,\mathrm{d}u \\ &\geqslant C_H\int_0^s m^{2H-2}\,\mathrm{d}m\int_0^m n^{H-\frac{3}{2}}dn\int_0^n u^{1-2H}(n-u)^{H-\frac{1}{2}}\,\mathrm{d}u \\ &= C_H\left(\int_0^1 x^{1-2H}(1-x)^{H-\frac{1}{2}}\,\mathrm{d}x\right)s^{2H}.\end{aligned}$$

Thus, estimate (A.4) follows from the inequality

$$s^{2H} \geqslant t^{2H} - (t-s)^{2H}$$

for all $0 < s \leqslant t$. □

Lemma A.5. *Let* $0 < H < \frac{1}{2}$*. Then the estimates*

$$(st)^{H-\frac{1}{2}} \int_0^s u^{1-2H}[(t-u)(s-u)]^{H-\frac{1}{2}}\,\mathrm{d}u \leqslant C_H G(t,s),$$

$$t^{H-\frac{1}{2}} \int_0^s u^{1-2H}(t-u)^{H-\frac{1}{2}}\,\mathrm{d}u \int_u^s (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}}\,\mathrm{d}m \leqslant C_H G(t,s),$$

$$s^{H-\frac{1}{2}} \int_0^s u^{1-2H}(s-u)^{H-\frac{1}{2}}\,\mathrm{d}u \int_u^t (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}}\,\mathrm{d}m \leqslant C_H G(t,s),$$

$$\int_0^s u^{1-2H}\,\mathrm{d}u \int_u^t \int_u^s (mn)^{H-\frac{3}{2}}[(m-u)(n-u)]^{H-\frac{1}{2}}\,\mathrm{d}m\,\mathrm{d}n \leqslant C_H G(t,s)$$

hold for all $0 \leqslant s \leqslant t \leqslant T$*, where* $G(t,s) = t^{2H} + s^{2H} - |t-s|^{2H}$.

Proof. Given $0 < H < \frac{1}{2}$. For the first estimate, we have

$$\begin{aligned}
(st)^{H-\frac{1}{2}} &\int_0^s u^{1-2H}[(t-u)(s-u)]^{H-\frac{1}{2}}\,\mathrm{d}u \\
&\leqslant \int_0^s [(t-u)(s-u)]^{H-\frac{1}{2}}\,\mathrm{d}u \leqslant \frac{1}{2}\int_0^s [(t-u)^{2H-1} + (s-u)^{2H-1}]\mathrm{d}u \\
&\leqslant C_H[t^{2H} + s^{2H} - (t-s)^{2H}].
\end{aligned}$$

For the second estimate, we have

$$\begin{aligned}
t^{H-\frac{1}{2}} &\int_0^s u^{1-2H}(t-u)^{H-\frac{1}{2}}\,\mathrm{d}u \int_u^s (m-u)^{H-\frac{1}{2}} m^{H-\frac{3}{2}}\,\mathrm{d}m \\
&= t^{H-\frac{1}{2}} \int_0^s m^{H-\frac{3}{2}}\,\mathrm{d}m \int_0^m u^{1-2H}(t-u)^{H-\frac{1}{2}}(m-u)^{H-\frac{1}{2}}\,\mathrm{d}u \\
&\leqslant t^{H-\frac{1}{2}} \int_0^s m^{H}\,\mathrm{d}m \int_0^m u^{1-2(H+\frac{1}{2})}[(t-u)(m-u)]^{(H+\frac{1}{2})-\frac{3}{2}}\,\mathrm{d}u \\
&= \alpha_{H+\frac{1}{2}} \int_0^s (t-m)^{2H-1}\,\mathrm{d}m \\
&\leqslant C_H[t^{2H} + s^{2H} - (t-s)^{2H}]
\end{aligned}$$

by lemma A.1. Similarly, one can get the third and fourth estimates. This completes the proof. □

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